# 10.2 图的术语和特殊图

*66. Show that if G is a bipartite simple graph with v vertices and e edges, then $e \leqslant v^2/4$.
Solution:
证二分简单图的 $e \leqslant v^2/4$

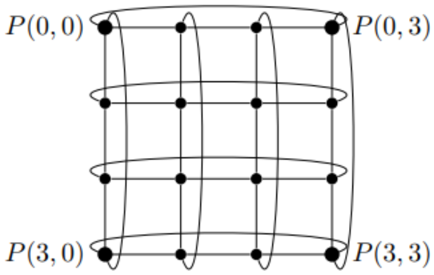
$v=v_1+v_2$;　握手定理 $\sum deg(v)=2e$;

**Bipartite Graph, $deg(v_1) \le v_1 * v_2$;$deg(v_2 2) \le v_2 * v_1$;**

**$v_1 * v_2 \le v/2 * v/2 = v^2/4$**

**e =** $\sum deg(v) /2 = (deg(v_1)+deg(v_2))/2$ **$\le v_1 * v_2$** $\leqslant v^2/4$

74. In a variant of a mesh network for interconnecting n = m2 processors, processor P (i, j )is connected to the four processors P ((i ± 1) mod m, j ) and P (i, (j ± 1) mod m), so that connections wrap around the edges of the mesh. Draw this variant of the mesh network for 16 processors.
Solution:
In addition to the connections shown in Figure 13, we need to make connections between P(i, 3) and P(i, 0) for each i, and between P(3, j) and P(0, j) for each j . The complete network is shown here. We can imagine this drawn on a torus.




# 10.3 图的表示和图同构

50. Suppose that G and H are isomorphic simple graphs. Show that their complementary graphs G and H are also isomorphic.

Solution：

since $G(V_1,E_1) \cong H(V_2,E_2)$,

exist a f: $V_1$->$V_2$ is bijection, all $(u,v) \in E_1$ iff $(f(u),f(v)) \in E_2$.

U-G($V_1$,~$E_1$), U-H($V_2$,~$E_2$), same f: $V_1$->$V_2$,

since an edge is in U-G iff it is not in G,

$(u,v) \in \sim E_1$ iff $(u,v) \notin E_1$ iff $(f(u),f(v)) \notin E_2$ iff $(f(u),f(v)) \in \sim E_2$

so all $(u,v) \in \sim E_1$ iff $(f(u),f(v)) \in \sim E_2$

Hence U-G($V_1$,~$E_1$) $\cong$ U-H($V_2$,~$E_2$)

*56. Show that if G is a self-complementary simple graph with v vertices, then v

$\equiv$ 0 or 1 (mod 4).

Solution：if G and U-G is isomorphism,named self-complementary.

Because G and U-G is isomorphism, $e_1=e_2$.

$E_1 \cap E_2=\varnothing$, so $|E_1 \cup E_2| = e_1+e_2 =2e_1$ .

since the union of the two graphs is Kn.

Kn: $e=n*(n-1)/2$. so $e_1=e_2=e/2=n*(n-1)/4$.

n must be integer, so $n*(n-1)=4m$, m is a integer.

=> n mod 4 =0, or 1.

*74. How many nonisomorphic directed simple graphs are there with n vertices, when n is a) 2? b) 3? c) 4?

Solution：(原书答案有误)

nonisomorphic directed simple graphs

n=2, s(n)=1(e0)+1(e1)+1(e2)=3;

n=3, s(n)=1+1+4+4+4+1+1=16;

n=4, s(n)=1+1+4+8+10+2+3+1+1=31;

## 10.4 图的连通性

*28. Show that every connected graph with n vertices has at least n − 1 edges.

Proof: 证明 n 点的连通图至少有 n-1 条边

We show this by induction on n. For n = 1 there is nothing to prove.

Now assume the inductive hypothesis, and let G be a connected graph with n +1 vertices and fewer than n edges, where $n \geqslant 1$. $\sum deg(v)=2e <2n <2(n+1)$; Therefore some vertex has degree less than 2. Since G is connected, this vertex is not isolated, so it must have degree 1.

Remove this vertex and its edge. Clearly the result is still connected,

and it has n vertices and fewer than n−1 edges,

contradicting the inductive hypothesis. Therefore the statement holds for G, and the proof is complete.

*36. Show that a vertex c in the connected simple graph G is a cut vertex if and only if there are vertices u and v, both different from c, such that every path between u and v passes through c.

Prove: 连通简单图的割点 c 充要条件是其它点路径必然经过 c.

if c is a cut vertex, Since the removal of c increases the number of components, there must be two vertices in different components. Then every path between these two vertices has to pass through c.

if u and v are as specified, then they must be in different components of the graph with c removed. Therefore the removal of c resulted in at least two components, so c is a cut vertex.

60. Show that the existence of a simple circuit of length k, where k is an integer greater than 2, is an invariant for graph isomorphism.

Proof：证明长度为 k 的简单回路是图形不变量

Suppose that f is an isomorphism from graph G to graph H. If G has a simple circuit of length k, say u1,u2,...,uk,u1. since each edge $u_iu_{i+1}$ (and $u_ku_1$) in G corresponds to an edge $f(u_i)f(u_{i+1})$ (and $f(u_k)f(u_1)$) in H.

Furthermore, since no edge was repeated in this circuit in G, no edge will be repeated when we use f to move over to H.

# 10.5 欧拉回路和哈密顿回路

*16. Show that a directed multigraph having no isolated vertices has an Euler circuit if and only if the graph is weakly connected and the in-degree and out-degree of each vertex are equal.

Proof: 有向图具有欧拉回路 iff 弱连通且 all $deg^+(v_i)=deg^-(v_i)$.

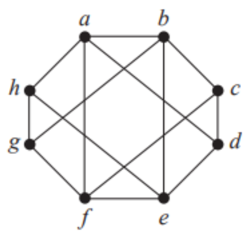
First suppose that the directed multigraph has an Euler circuit. the graph must be strongly connected. as the circuit passes through a vertex, it adds one to the count of both the in-degree and the out-degree.

Conversely, suppose that the graph meets the conditions stated. Then we can proceed as in the proof of Theorem 1 and construct an Euler circuit.

# 10.6 图的最短路径问题

# 10.7 平面图性质和判定

8. determine whether the given graph is planar. If so, draw it so that no edges cross.



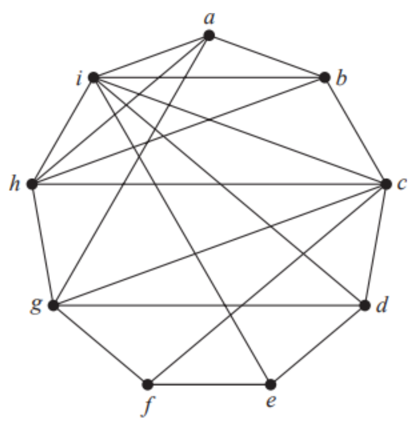

Solution:

不是平面图，判定理由只能是包含 $K_{3,3}$ 或 $K_5$ 同胚子图。

V1={a,c,e}，对应 V2={b,d,f}，可以构成 $K_{3,3}$ 子图.

# 10.8 图着色数

10. find the chromatic number of the given graph.




Solution:
Since vertices b, c, h, and i form a $K_4$ , at least 4 colors are required. A coloring using only 4 colors is to let a and c be red; b, d, and f , blue; g and i, green; and e and h, yellow.

23. Find the edge chromatic numbers of

a) $C_n$, where n ⩾ 3.

b) $W_n$, where n ⩾ 3.

Solution:
a) 2 if n is even, 3 if n is odd.
环图中每个点关联两条边，所以边交替着色，但 2 色要求奇数个边；否则 3 色。
b) n
轮图的中继点关联 n 条边，所以需要 n 色，剩余环图着色 2 或 3，但要避开三角形中两条中继边颜色。因为 n 一定>2,所以环路部分的着色在 n 之内选择足够。

## 补充图着色多项式 13、14、15、16;

*In Exercises 13 through 16 (Figures 8.111 through 8.114), find the chromatic polynomial $P_G$ for the given graph and use $P_G$ to find $\chi(G)$.*

**13.**

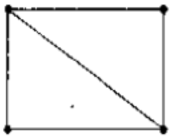


Figure 8.111

**14.**

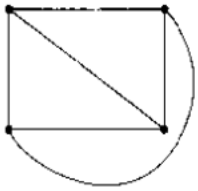


Figure 8.112

**15.**

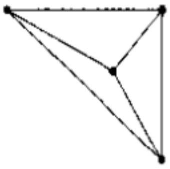


Figure 8.113

**16.**

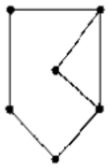


Figure 8.114

Solution:

**13.** $P_G(x) = x(x-1)(x-2)^2$; $\chi(G) = 3$.

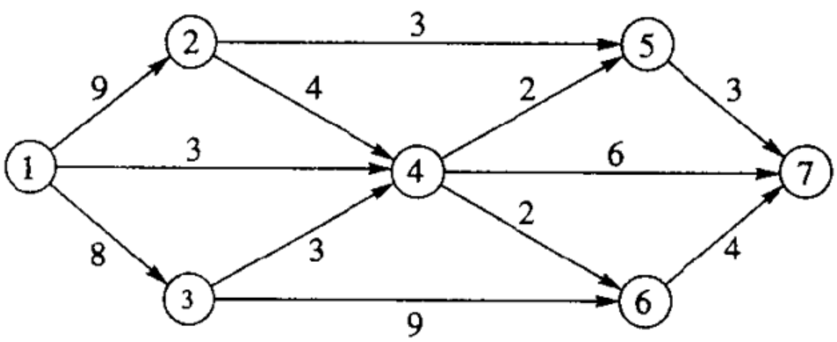
**15.** $P_G(x) = x(x-1)(x-2)(x-3)$; $\chi(G) = 4$.

14.$P_G(x)=x(x-1)(x-2)(x-3)$. X(G)=4.

16. $P_G(x)=(x-2)P_{C_5}(x)=x(x-1)(x-2)(x^3-4x^2+6x-4)$

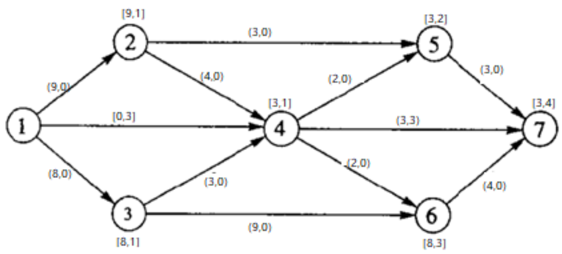
## <离散数学结构>书上的最大流量算法题 10,14@P305;

10. find a maximum flow in the given network by using the labeling algorithm.

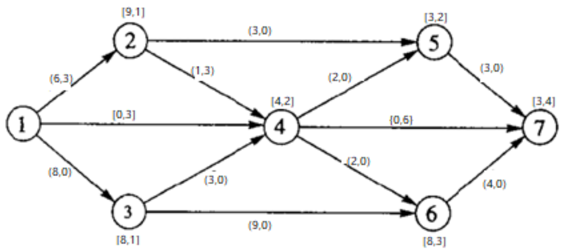



Solution: 注意增广路径的发现顺序。

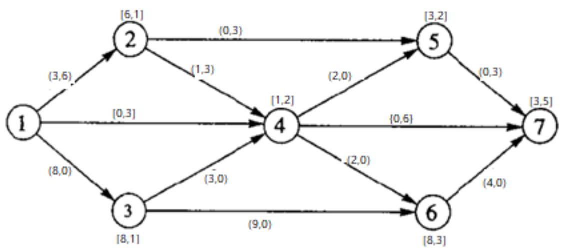
N1={2,3,4} N2={5,6,7}, path 1-4-7, -3.



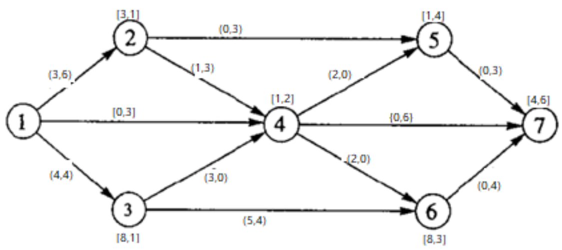

N1={2,3} N2={4,5,6} N3={7}, path 1-2-4-7, -3



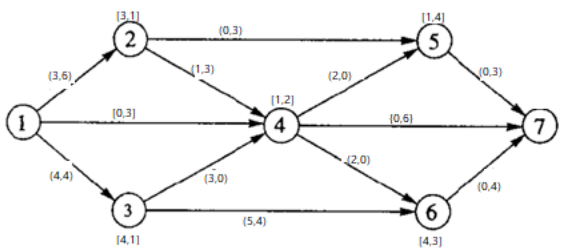

N1={2,3} N2={4,5,6} N3={7}, path 1-2-5-7, -3



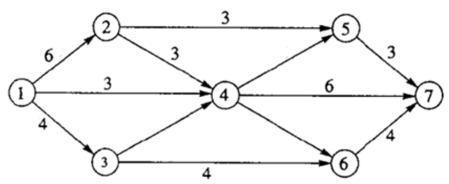

N1={2,3} N2={4,6} N3={5,7}, path 1-3-6-7, -4



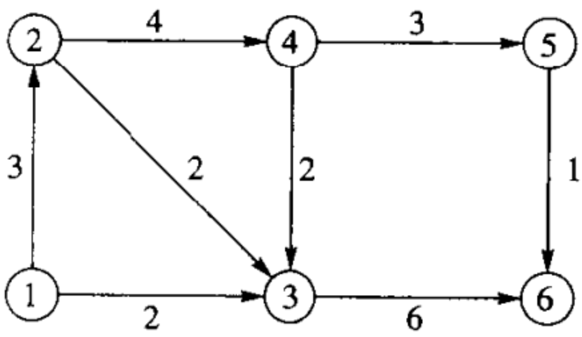

N1={2,3} N2={4,6} N3={5}    stop. maxflow=13




最大流设计图为

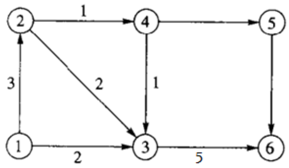



14. find a maximum flow in the given network by using the labeling algorithm.




Solution: minimal cut K={(1,2),(1,3)}, maxflow=5.
注意增广路径的发现顺序。

N1={2,3} N2={4,6}, path 1-3-6, -2.
N1={2} N2={3,4} N3={6}, path 1-2-3-6, -2
N1={2} N2={4} N3={3,5} N4={6}, path 1-2-4-3-6, -1
N1={} stop. maxflow=5

最大流设计图为 


# 递推关系 8.1

8.1：14. a) Find a recurrence relation for the number of ternary strings of length n that contain two consecutive 0s.

b) What are the initial conditions?

c) How many ternary strings of length six contain two consecutive 0s?

解答：

a) Let $a_n$ be the number of ternary strings that contain two consecutive 0's.
To construct such a string we could start with either a 1 or a 2 and follow with a string containing two consecutive 0's (and this can be done in $2a_{n-1}$ ways), or we could start with 01 or 02 and follow with a string containing two consecutive 0's (and this can be done in $2a_{n-2}$ ways), we could start with 00 and follow with any ternary string of length n − 2 (of which there are clearly $3^{n-2}$ ). Therefore the recurrence relation, valid for all n ⩾ 2, is $a_n = 2a_{n-1} + 2a_{n-2} +$

$3^{n-2}$.

注意:　长度 n-2 的任意 3 进制串有 $\mathbf{3}^{n-2}$ ，加 00 都为合格串。

长度 n-1 合格串后面加 0，会和"n-2 串+00"处理重复，故去掉，只留+1/+2 两种;
长度 n-2 合格串后面加 01，02，会和长度 n-1 末尾 0 的合格串处理重复，故去掉;
长度 n-2 合格串后面加 1 或 2，会形成 n-1 合格串，重复去掉，只剩再加 1 或 2 两种;

b) Clearly $a_0 = a_1 = 0$.
c) We will compute $a_2$ through $a_6$ using the recurrence relation:
$a_2 = 2\ a_1 + 2\ a_0 + 3^0 = 2 \cdot 0 + 2 \cdot 0 + 1 = 1$
$a_3 = 2a_2 + 2a_1 + 3^1 = 2 \cdot 1 + 2 \cdot 0 + 3 = 5$
$a_4 = 2a_3 + 2a_2 + 3^2 = 2 \cdot 5 + 2 \cdot 1 + 9 = 21$
$a_5 = 2\ a_4 + 2a_3 + 3^3 = 2 \cdot 21 + 2 \cdot 5 + 27 = 79$
$a_6 = 2a_5 + 2a_4 + 3^4 = 2 \cdot 79 + 2 \cdot 21 + 81 = 281$
Thus there are 281 bit strings of length 6 containing two consecutive 0's.

# 8.2 解方程求解齐次递推关系;

*10. Prove Theorem 2.

## Proof of Theorem 2

$c_1 a_{n-1} + c_2 a_{n-2}$
$= c_1(\alpha_1 r_0^{n-1} + \alpha_2 (n-1) r_0^{n-1}) + c_2(\alpha_1 r_0^{n-2} + \alpha_2 (n-2) r_0^{n-2})$
$= \alpha_1 r_0^{n-2}(c_1 r_0 + c_2) + \alpha_2 n r_0^{n-2}(c_1 r_0 + c_2) - \alpha_2 r_0^{n-2}(c_1 r_0 + 2c_2)$
$= \alpha_1 r_0^{n-2}\ r_0^2 + \alpha_2 n r_0^{n-2}\ r_0^2 - \alpha_2 r_0^{n-2}(c_1 r_0 + 2c_2)$
Because $r_0$ is repeated root, $b^2 - 4ac = {c_1}^2 + 4c_2 = 0$, $r_0 = \frac{-b}{2a} = \frac{c_1}{2}$.

$$(c_1 r_0 + 2c_2) = c_1 \frac{c_1}{2} + 2c_2 = \frac{{c_1}^2 + 4c_2}{2} = 0$$

$= \alpha_1 r_0^n + \alpha_2 n r_0^n$
$= a_n$
$a_0 = C_0 = \alpha_1$
$a_1 = C_1 = \alpha_1 r_0 + \alpha_2 r_0$
$\alpha_1 = C_0,\ \alpha_2 = \frac{C_1 - C_0 r_0}{r_0}$

42. Show that if $a_n = a_{n-1} + a_{n-2}, a_0 = s, a_1 = t$, where s and t are constants, then $a_n = sf_{n-1} + tf_n$ for all positive integers n.

Proof: We can prove this by induction on n.

base step : If n = 1, then the assertion is $a_1 = s \cdot f_0 + t \cdot f_1 = s \cdot 0 + t \cdot 1 = t$, which is given;
and if n = 2, then the assertion is $a_2 = s \cdot f_1 + t \cdot f_2 = s \cdot 1 + t \cdot 1 = s + t$,
which is true, since $a_2 = a_1 + a_0 = t + s$.

inductive step: we assume the inductive hypothesis, that the statement is true for values less than n.

Then $a_n = a_{n-1} + a_{n-2} = (sf_{n-2} + tf_{n-1}) + (sf_{n-3} + tf_{n-2}) = s(f_{n-2} + f_{n-3}) + t(f_{n-1} + f_{n-2})$

$= sf_{n-1} + tf_n$ , as desired.

46. Suppose that there are two goats on an island initially. The number of goats on the island doubles every year by natural reproduction, and some goats are either added or removed each year.

a) Construct a recurrence relation for the number of goats on the island at the start of the nth year, assuming that during each year an extra 100 goats are put on the island.

b) Solve the recurrence relation from part (a) to find the number of goats on the island at the start of the nth year.

c) Construct a recurrence relation for the number of goats on the island at the start of the nth year, assuming that n goats are removed during the nth year for each n ⩾ 3.

d) Solve the recurrence relation in part (c) for the number of goats on the island at the start of the nth year.

Solution:

Let $a_n$ represent the number of goats on the island at the start of the $n^{th}$ year.

a) The initial condition is $a_1 = 2$; we are told that at the beginning of the first year there are two goats. During each subsequent year (year n, with n ⩾ 2 ), the goats who were on the island the year before (year n − 1 ) double in number, and an extra 100 goats are added in.

So $a_n = 2a_{n-1} + 100$.

b) The associated homogeneous recurrence relation is $a_n=2a_{n-1}$ , whose solution is $a^{(h)}_n = \alpha 2^n$ .

The particular solution is a polynomial of degree 0, namely a constant, $a_n$ = c.

Plugging this into the recurrence relation gives c = 2c + 100, whence c = −100.

So the particular solution is $a^{(p)}_n = -100$ and the general solution is $a_n = \alpha 2^n - 100$.

Plugging in the initial condition and solving for α gives us 2 = 2α − 100, then α = 51.

Hence the desired formula is $a_n = 51 \cdot 2^n - 100$. There are $51 \cdot 2^n - 100$ goats on the island at the start of the $n^{th}$ year.

c) We are told that $a_1 = 2$, but that is not the relevant initial condition. Instead, since the first two years are special (no goats are removed), the relevant initial condition is $a_2 = 4$. During each subsequent year (year n, with n ⩾ 3 ), the goats who were on the island the year before (year n − 1 ) double in number, and n goats are removed. So $a_n=2a_{n-1} - n$. (We assume that the removal occurs after the doubling has occurred; if we assume that the removal takes place first, then we'd have to write a $a_n=2(a_{n-1} - n) = 2a_{n-1} - 2n$.)

d) The associated homogeneous recurrence relation is $a_n=2a_{n-1}$ , whose solution is $a^{(h)}_n = \alpha 2^n$ .

The particular solution is a polynomial of degree 1, say $a_n$ = cn + d. Plugging this into the recurrence relation and grouping like terms gives (−c + 1)n + (2c − d) = 0, whence c = 1 and d = 2. So the particular solution is $a^{(p)}_n = n + 2$ and the general solution is $a_n = \alpha 2^n + n + 2$.

Plugging in the initial condition $a_2 = 4$ and solving for α gives us 4 = 4α + 4, or α = 0. Hence the desired formula is simply $a_n = n + 2$ for all n ⩾ 2 (and $a_1 = 2$ ). There are n + 2 goats on the island at the start of the n th year, for all n ⩾ 2.

# 分治算法 8.3

14. Suppose that there are $n = 2^k$ teams in an elimination tournament, where there are n/2 games in the first round, with the $n/2 = 2^{k-1}$ winners playing in the

second round, and so on. Develop a recurrence relation for the number of rounds in the tournament.

Solution:

If there is only one team, then no rounds are needed, so the base case is *$R(1) = 0$*. Since it takes one round to cut the number of teams in half, we have $R(n) = 1 + R(n/2)$.

36. Find f (n) when $n = 2^k$ , where f satisfies the recurrence relation $f(n) = 8f(n/2) + n^2$ with $f(1) = 1$.

Solution:

From Exercise 31 (note that here a = 8, b = 2, c = 1, and d = 2 ) we have $f(n) = -n^2 + 2n^{\log 8}$

$= -n^2 + 2n^3$.

# 生成函数 8.4

16. Use generating functions to find the number of ways to choose a dozen bagels from three varieties—egg, salty, and plain—if at least two bagels of each kind but no more than three salty bagels are chosen.

Solution: Use generating functions to find the number of ways to choose a dozen bagels from three varieties—egg, salty, and plain—if at least two bagels of each kind but no more than three salty bagels are chosen.

$x_1+x_2+x_3=12$, $x_1\geqslant 2$, $2\leqslant x_2\leqslant 3$, $x_3\geqslant 2$.

$(x^2 + x^3 + x^4 + \cdots)(x^2 + x^3)(x^2 + x^3 + x^4 + \cdots)$,find the coefficient of $x^{12}$

$=x^6(1+x+x^2 + x^3 + x^4 + \cdots)^2(1+x)$

$=x^6(1+x)/(1-x)^2 =x^6(1/(1-x)^2 +x/(1-x)^2)$ ,

find the coefficient $a_6+a_5$ of $1/(1-x)^2$

$a_6=7$,$a_5=6$, answer: $a_6+a_5=13$.

36. Use generating functions to solve the recurrence relation $a_k = a_{k-1} + 2a_{k-2} + 2^k$ with initial conditions $a_0 = 4$ and $a_1 = 12$.

Solution:

$$(1)\, G(x) = \sum_{k=0}^{\infty} a_k x^k$$

$$(2)\ G(x) - xG(x) - 2x^2G(x) = \sum_{k=0}^{\infty} a_k x^k - x\sum_{k=0}^{\infty} a_k x^k - 2x^2 \sum_{k=0}^{\infty} a_k x^k$$

$$= \sum_{k=0}^{\infty} a_k x^k - \sum_{k=1}^{\infty} a_{k-1} x^k - \sum_{k=2}^{\infty} 2a_{k-2} x^k$$

$$= (a_0 + a_1 x) + (-a_0 x) + \sum_{k=2}^{\infty} (a_k - a_{k-1} - 2a_{k-2}) x^k$$

$$= a_0 + a_1 x - a_0 x + \sum_{k=2}^{\infty} 2^k x^k$$

$$= 4 + 12x - 4x + (\sum_{k=0}^{\infty} 2^k x^k - 2^0 x^0 - 2^1 x^1)$$

$$= 4 + 8x + \frac{1}{1-2x} - 1 - 2x = \frac{4-12x^2}{1-2x}$$

$$(3)\ G(x) = \frac{\frac{4-12x^2}{1-2x}}{1-x-2x^2} = \frac{4-12x^2}{(1-2x)^2(1+x)} = \frac{-8/9}{1+x} + \frac{38/9}{1-2x} + \frac{2/3}{(1-2x)^2}$$

$$= \sum_{k=0}^{\infty} (-8/9)(-1)^k x^k + \sum_{k=0}^{\infty} (38/9) 2^k x^k + (2/3)\sum_{k=0}^{\infty} (\sum_{j=0}^{k} 2^j 2^{k-j}) x^k$$

$$= \sum_{k=0}^{\infty} ((\frac{-8}{9})(-1)^k + (\frac{38}{9})2^k + \frac{2}{3}(2^k)(k+1)) x^k$$

$$So\ \mathrm{a_k} = (-8/9)(-1)^k + (38/9)2^k + (2/3)(k+1)2^k$$