

1. [8 points] For each of these relations, decide whether it is reflexive, whether it is symmetric, whether it is antisymmetric, and whether it is transitive.
 每问 2 分, 每问包含 4 个结果, 错一个扣 0.5 分; 本题最终扣分向下取整。
- a) "divides" relation on the set of nonnegative integers.
Not Reflexive, not symmetric, antisymmetric, transitive.
- b) The inverse relation of $R = \{(2, 5), (5, 2), (2, 2), (5, 5)\}$ on the set $\{1, 2, 3, 4, 5\}$.
Not Reflexive, symmetric, not antisymmetric, transitive.
- c) The complementary relation of $R = \{(1, 3), (2, 3), (3, 1), (3, 2)\}$ on the set $\{1, 2, 3\}$.
Reflexive, symmetric, not antisymmetric, transitive.
- d) The composite of "less than" relation on the set of integers and "greater than" relation on the set of integers.
Reflexive, symmetric, not antisymmetric, transitive.

Example: Let $S = \{1, 2, 3, \dots\}$, then $R_1 \circ R_2 =$

0	1	1	0	0	0	1	1	0
0	0	1	1	0	0	1	1	0
0	0	0	1	1	0	0	0	0

2. [8 points] Use Warshall's algorithm to find the transitive closure of R on $\{1, 2, 3, 4, 5\}$ where $R = \{(1, 2), (1, 3), (2, 2), (2, 3), (2, 4), (3, 2), (3, 3), (3, 4), (3, 5), (4, 5), (5, 1)\}$.

$$y'' + P(x)y' + Q(x)y = R(x)$$

常数

$$r^2 + Pr + Q = 0$$

$$x^{2m} (x^m) y_1$$

$(2m) \neq \text{整数} \rightarrow y_2$

当 n 为整数时

$$J_n(x) =$$

$$J_n(x) =$$

Example: Let $S=\{1,2,3,\dots\}$, then $R1^{\circ}R2=$

2. [8 points] Use Warshall's algorithm to find the transitive closure of R on $\{1,2,3,4,5\}$ where $R=\{(1,2), (1,3), (2,2), (2,3), (2,4), (3,2), (3,3), (3,4), (3,5), (4,5), (5,1)\}$.

$$w_0 = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}, w_1 = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 \end{bmatrix}, w_2 = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix}$$

Page 1 of 7

$$w_3 = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}, w_4 = w_3, w_5 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$$y' + P(x)y$$

$$y' + Py' -$$

$$r^2 + Pr + Q$$

$$x^{2M} (x^m)$$

$$(2M) \pm \text{integer}$$

当M为整数

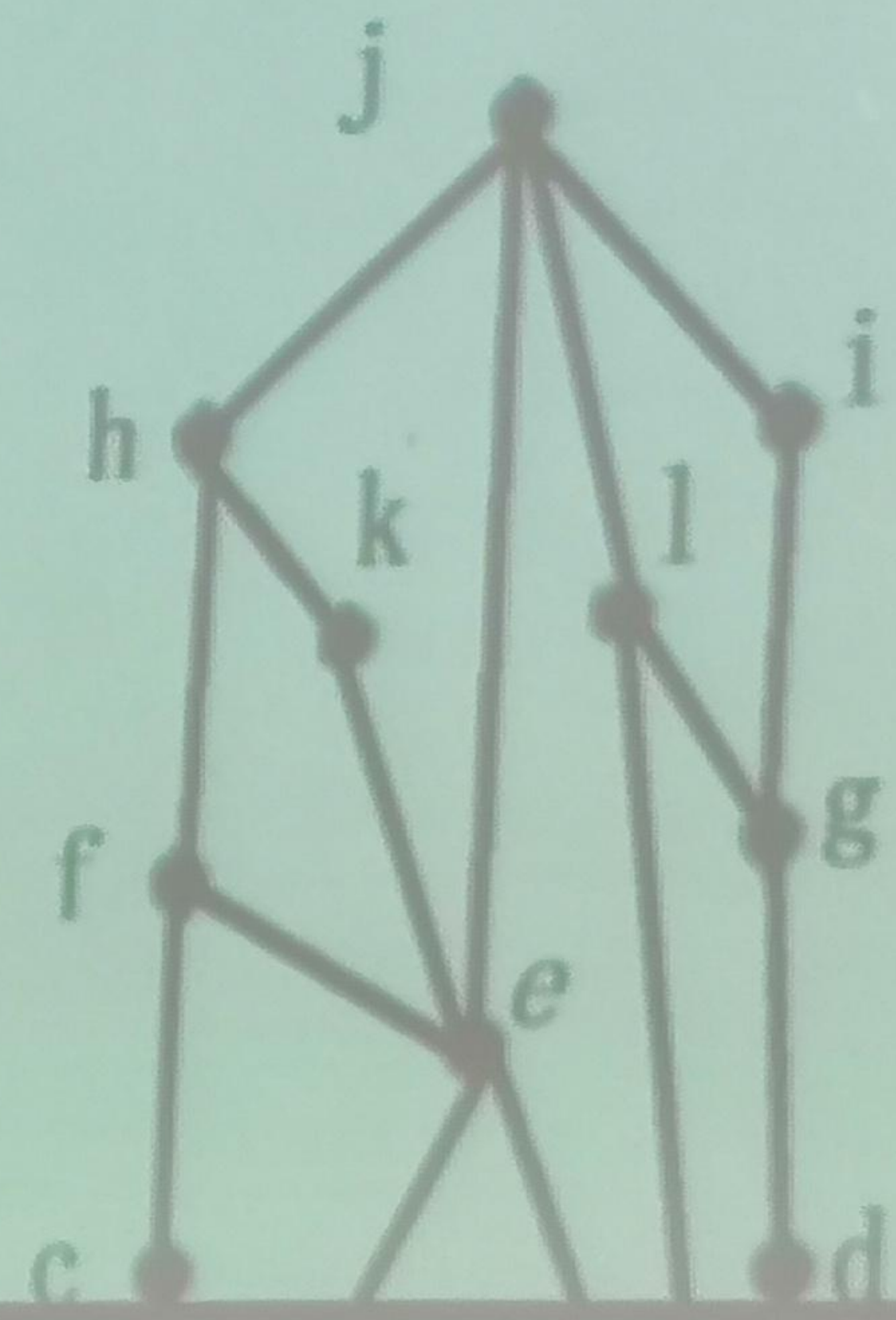
定义 $\ln(x)$

全对得4分, w5中错2个元素扣1分, 扣完4分为止。

3. [8 points] Let R_1 is an equivalence relation produced by the partition $A_1=\{a,b\}$, $A_2=\{c,d\}$, and $A_3=\{e,f\}$ of $S=\{a,b,c,d,e,f\}$, R_2 is another equivalence relation produced by the partition $A_1=\{a,c\}$, $A_2=\{b,d\}$, $A_3=\{e\}$, and $A_4=\{f\}$ of $S=\{a,b,c,d,e,f\}$. List the ordered pairs of relation (1) $R_1 \cap R_2$ and (2) $R_1 \oplus R_2$.

(1) $R_1 \cap R_2 = \{(a,a), (b,b), (c,c), (d,d), (e,e), (f,f)\}$
 (2) $R_1 \oplus R_2 = \{(a,b), (b,a), (c,d), (d,c), (e,f), (f,e), (a,c), (c,a), (b,d), (d,b)\}$.
 每问4分, 错2个元素扣1分, 扣完4分为止。

4. [12 points] Answer these questions for the partial order represented by this Hasse diagram.



Handwritten notes on the chalkboard:

$$y'' + P(x)y' + Q(x)y = R(x)$$

常数

$$r^2 + Pr + Q = 0$$

x^{2m} (circled), x^m (circled)

$2m \pm \text{整数}$ (circled)

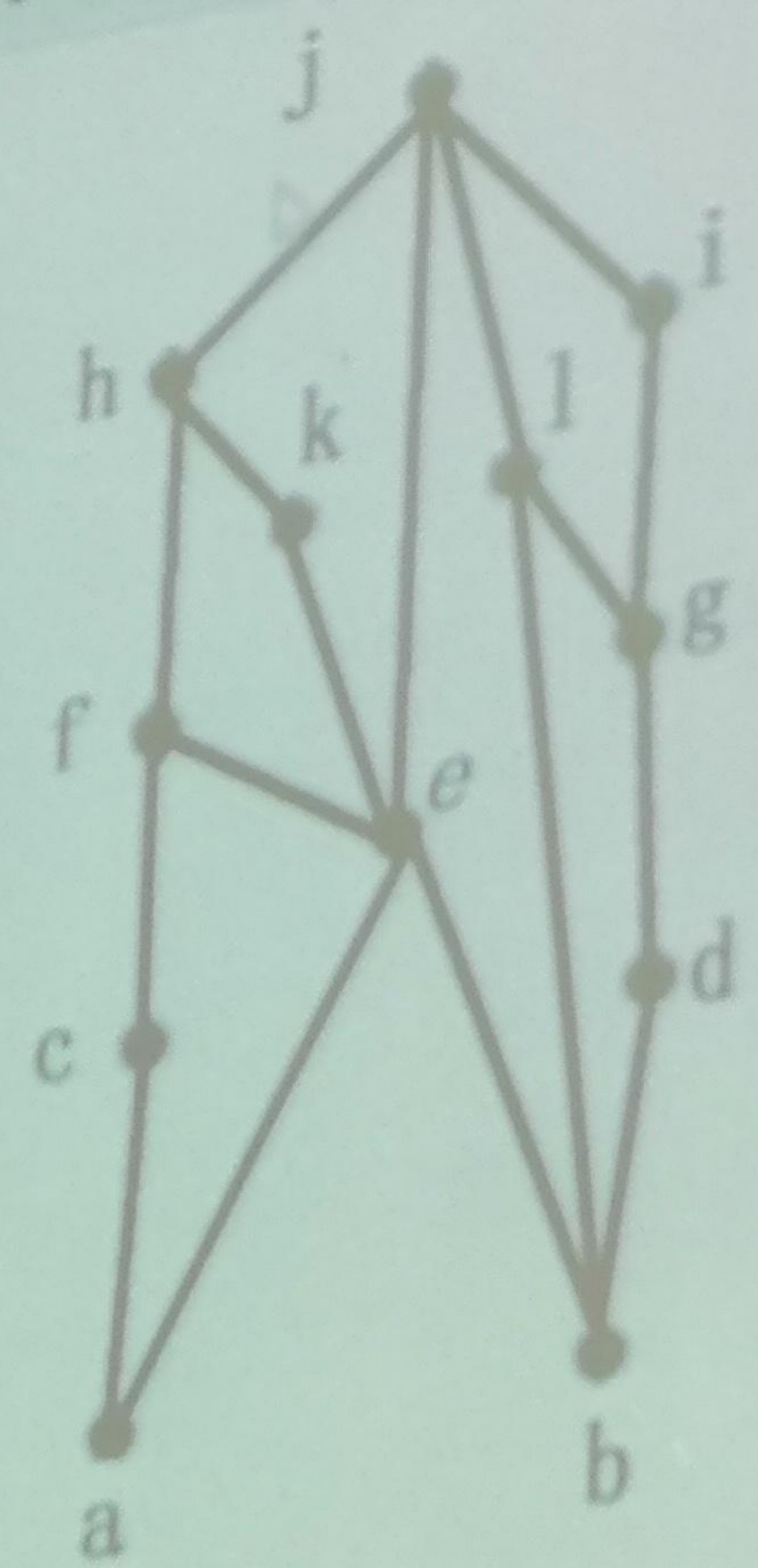
Handwritten notes on the chalkboard:

当 m 为整数 n 时

定义 $y_m(x)$

$y_n(x)$





- (1) Find all maximal elements.
 $\{j\}$ 正确得 1 分
- (2) Find all minimal elements.
 $\{a, b\}$ 正确得 1 分
- (3) Is there a greatest element?
 Yes 正确得 1 分

$$y' + P(x)y' + \dots$$

$$y' + Py' + \frac{q}{y}$$

常数

$$(2M) +$$

(4) Is there a least element?

No 正确得 1 分

(5) Find all upper bounds of $\{a, b, c\}$.

$\{f, h, j\}$ 正确得 2 分, 写错或漏一个扣 1 分

(6) Find the least upper bound of $\{e, f, g\}$, if it exists.

j 正确得 2 分, 答不存在的扣 2 分, 写错元素的扣 1 分。

(7) Find all lower bounds of $\{h, j, k\}$.

$\{k, e, a, b\}$ 正确得 2 分, 写错或漏一个扣 1 分。

(8) Find the greatest lower bound of $\{i, j, l\}$, if it exists.

g 正确得 2 分, 答不存在的扣 2 分, 写错元素的扣 1 分。

5. [10 points] Let Q be the set of rational numbers and define $a * b = a + b - ab$

(a) Is $(Q, *)$ a monoid? Justify your answer.

(b) If $(Q, *)$ a monoid, which elements of Q have an inverse?

答: (1)

由于 a, b 是实数, 所以 $ab = a + b - ab$ 也是实数, 运算闭合。 (1 分)

$$a * (b * c) = a * (b + c - bc) = a + (b + c - bc) - a(b + c - bc)$$

$$y' + P(x)y'$$
$$y' + Py' +$$

5. [10 points] Let Q be the set of rational numbers and define $a * b = a + b - ab$
- (a) Is $(Q, *)$ a monoid? Justify your answer.
- (b) If $(Q, *)$ a monoid, which elements of Q have an inverse?

答: (1)

由于 a, b 是实数, 所以 $ab = a + b - ab$ 也是实数, 运算闭合. (1分)

$$a * (b * c) = a * (b + c - bc) = a + (b + c - bc) - a(b + c - bc) \\ = a + b + c - bc - ab - ac + abc$$

$$(a * b) * c = (a + b - ab) * c = a + b - ab + c - (a + b - ab)c \\ = a + b + c - ab - ac - bc + abc$$

(3分)

因此 $*$ 是可结合的.

由于 $a * 0 = 0 * a = a$, 所以 0 是单位元, $(Q, *)$ 是独异点. (1分)

(2)

如果 $a \neq 1$,

$$a * \frac{-a}{1-a} = a + \frac{-a}{1-a} - a \times \frac{-a}{1-a} = 0$$

(3分)

所以, Q 中除 1 以外都有逆元.

(2分)

6. [10 points] Let G be a group and K is a subgroup of G . Show that for any element a, b in G , either $aK = bK$ or $aK \cap bK = \emptyset$.

解法 1—

证明: 思路是构造一个等价关系, 使得 K 的每个左陪集都是其中的等价类

$$y' + P(x)y'$$

$$y' + Py' -$$

常数

$$r^2 + Pr + \frac{1}{r} =$$

$$2M \neq \text{整数} \rightarrow$$

当 M 为奇数

所以, Q 中除 1 以外都有逆元.

6. [10 points] Let G be a group and K is a subgroup of G . Show that for any element a, b in G , either $aK = bK$ or $aK \cap bK = \emptyset$. (2分)

解法 1-

证明: 思路是构造一个等价关系, 使得 K 的每个左陪集都是其中的等价类.
1. 构造一个二元关系 $R = \{aRb \mid a^{-1}b \in K\}$. (2分)

下面证明它是一个等价关系.

自反性: $\forall x \in G, x^{-1}x = e \in K, \Rightarrow xRx$. (1分)

对称性: $\forall x, y \in G, xRy \Rightarrow x^{-1}y \in K, \text{ so } y^{-1}x = (x^{-1}y)^{-1} \in K, yRx$. (1分)

传递性: $\forall x, y, z \in G, (xRy \text{ and } yRz) \Rightarrow x^{-1}y \in K \text{ and } y^{-1}z \in K$.

$x^{-1}z = x^{-1}y \cdot y^{-1}z \in K, xRz$. (1分)

2. 证明 R 的等价类是 K 的左陪集.

$$y'' + P(x)y' + Q(x)y = R(x)$$

常数

$$y'' + Py' + Qy = 0$$

$$2M \neq \dots$$

常数

$$r^2 + pr + q = 0$$

$$\frac{X^{2M}}{X^m} = X^m \rightarrow y_2$$

当 n 为整数时

家

$b \in aK$, 当且仅当 $a^{-1} * b \in K$.

证: $b \in aK$, 当且仅当存在 $k \in K$, 使得 $b = a * k$; (1分)

$\Rightarrow a^{-1} * b = k \in K$, 因此当且仅当 $a^{-1} * b \in K$. (1分)

因此, R 的 $[a] = aK$.

3. 证明 K 的所有左陪集, 或者 $aK = bK$, 或者 $aK \cap bK = \emptyset$.

因为 $aK = [a]$, $bK = [b]$,

根据等价类的性质可知, $[a] \cap [b] \neq \emptyset$ iff $[a] = [b]$.

所以 或者 $aK = bK$, 或者 $aK \cap bK = \emptyset$. (2分)

解法 2-

$$y'' + P(x)y'$$
$$y' + Py' -$$

常数

$$r^2 + Pr + Q$$

$$x^{2M} \text{ (X)}$$
$$(2M) + \text{整数}$$

二阶

当 M 为整数

定义 γ_m

γ_n

8. [12 points] Let Z be the set of all integers and let $+$ be the binary operation of addition on Z . Let $B = \{0, 1\}$, and let $+_2$ be the operation defined on B as follows

$+$	0	1
0	0	1
1	1	0

. Let f be a function from group $(Z, +)$ to $(B, +_2)$.
 (a) Prove that f defined by $f(x) = x \pmod{2}$ is homomorphism from Z to B .

证: (4分)

$$f(x+y) = (x+y) \pmod{2} = (x \pmod{2} + y \pmod{2}) \pmod{2} = (f(x) + f(y)) \pmod{2} \\ = f(x) +_2 f(y)$$

所以 f defined by $f(x) = x \pmod{2}$ is homomorphism from Z to B

(b) Find $\ker(f)$. (4分)

$$\ker(f) = \{2x | x \in Z\}$$

$$y'' + P(x)y' + Q(x)y = R(x)$$

常数

$$y^2 + Py + Q = 0$$

$2M \pm \text{整数}$

当 M 为整数时

定义 $\lim_{n \rightarrow \infty} x_n$

(c) Write the operation table of quotient group $Z/\ker(f)$. (4 分)

\oplus	[0]	[1]
[0]	[0]	[1]
[1]	[1]	[0]

(4 分, 运算表正确可得分, 错一个扣 1 分, 扣完为止)

9. [8 points] Let $H = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ be a parity check matrix.

$$y'' + \underline{P(x)} y' + \underline{Q(x)} y =$$

常数

$$r^2 + Pr + Q = 0 \quad r_1, r_2$$

$$x^{2M} (x^m)^{y_1} + \dots + (2M) + \dots \rightarrow y_2$$

当 n 为整数时

$$J_n(x) = \frac{J_{n+1}(x)}{x}$$

$$J_n(x) = \lim_{x \rightarrow 0} \dots$$

9. [8 points] Let $H = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ be a parity check matrix.

(a) Determine the $(3,7)$ group code $e_H: B^3 \rightarrow B^7$.

(6分, 8个结果错一个扣 1分, 扣完为止)

$$B^3 * H = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} * \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

$$y'' + P(x)y' + Q(x)y = R(x)$$

常数

$$r^2 + Pr + Q = 0$$

$$x^{2M} (x^m) y_1$$

$(2M) \pm \text{整数} \rightarrow y_2$

当 M 为整数 n 时

定义 $J_n(x) = \dots$

(a) Determine the (3,7) group code $e_H: B^3 \rightarrow B^7$.

(6分, 8个结果错一个扣 1分, 扣完为止)

$$B^3 * H = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} * \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}$$

或者写成如下形式:

$e(000)=0000000$, $e(001)=0011011$,
 $e(010)=0101101$, $e(011)=0110110$,
 $e(100)=1001110$, $e(101)=1010101$,
 $e(110)=1100011$, $e(111)=1111000$.

(b) How many errors will a detect?

$$y' + P(x)y'$$

$$y' + Py' +$$

常数

$$r^2 + Pr + \frac{1}{2} =$$

$$x^{2M} (x^m)$$

$$(2M) \neq \text{整数}$$

当时

当M为整数

定义 $\lim_{n \rightarrow \infty} x$

$\lim_{n \rightarrow \infty} x$

9. [8 points] Consider the (3,6) group encoding function $e: B^3 \rightarrow B^6$ defined by

$e(000)=000000$, $e(001)=001011$, $e(010)=010101$,
 $e(011)=011110$, $e(100)=100110$, $e(101)=101101$,
 $e(110)=110011$, $e(111)=111000$.

(a) How many errors will (e, d) correct?

Because the minimal distance of e is 3, (e, d) can correct 1 error.

(2分, 答对结果数即可得分, 结果错但最小距离算对了可得1分)

$$y'' + P$$
$$y' + P$$

$$r^2 + P$$

$$x^2 + P$$

$$(2M) \pm \dots$$

$$2M$$

$$2M$$

$$2M$$

000000	001011	010101	011110	100110	101101	110011	111000
000001	001010	010100	011111	100111	101100	110010	111001
000010	001001	010111	011100	100100	101111	110001	111010
000100	001111	010001	011010	100010	101001	110111	111100
001000	000011	011101	010110	101110	100101	111011	110000
010000	011011	000101	001110	110110	111101	100011	101000
100000	101011	110101	111110	000110	001101	010011	011000
001100	000111	011001	010010	101010	100001	111111	110100

(6分, 8个结果错一个扣 1分, 扣完为止)

The coset leaders for N is {000000, 000001, 000010, 000100, 001000, 010000, 100000, 001100 / 010010 / 100001}. 注意最后一个陪集头有3种写法都对

$$y'' + P$$

$$y' + P$$

$$r^2 + P$$

$$(2M) \pm \dots$$

当时
当M为

定义

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别要

理榜

10. [8 points] Let $m=3$, $n=6$, $H = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ be a parity check matrix.

(a) Compute the syndrome for the coset leaders $\{000000, 000001, 000010, 000100, 001000, 010000, 100000, 000110\}$ for $N = e_H(B^3)$.

(5分, 8对结果错一个扣 0.5分, 最终扣分若有小数则向下取整)

Syndrome	Coset leader
000	000000
001	000001
010	000010
100	000100

$$y'' + P(x)y' + Q(x)y = R(x)$$

常数

$$r^2 + Pr + Q = 0$$

$$x^{2M} \circledast x^m y_1$$

$(2M) \pm \text{整数} \rightarrow y_2$

当时 n 为整数时

$$Y_n(x) =$$

$$Y_n(x) =$$

function associated with e_H by using the coset leaders and their

syndromes from (a).

a) 100110 b) 011011 c) 110001

(3分, 错一个结果扣1分。)

$d(100110)=100$, $d(011011)=001$, $d(110001)=111$.

Because $100110 * H = 001$, so coset leader is 000001,

$100110 \oplus 000001 = 100111$, so $b=100$.

Similar, $011011 * H = 101$, so coset leader is 010000,

$011011 \oplus 010000 = 001011$, so $b=001$.

$110001 * H = 011$, so coset leader is 001000, $110001 \oplus 001000 = 111001$, so $b=111$.

$$y'' + P(x)y' + Q(x)y = R(x)$$

$$y' + Py' + Qy = R$$

常数

$$r^2 + Pr + Q = 0$$

$$\frac{x^{2M} (x^m)^{y_1}}{(2M) \pm \text{整数}} \rightarrow y_2$$

当 M 为整数时

$$J_n(x) = \frac{J_{n+1}(x)}{x}$$

$$Y_n(x) = \lim_{x \rightarrow 0}$$