1. [8 points] For each of these relations, decide whether it is reflexive, whether it is symmetric, whether it is antisymmetric, and whether it is transitive.

每问 2 分，每问包含 4 个结果，错一个扣 0.5 分；本题最终扣分向下取整。

a) "divides" relation on the set of nonnegative integers.

Not Reflexive, not symmetric, antisymmeytic, transitive.

b) The inverse relation of R={(2, 5), (5, 2),(2,2),(5,5)} on the set {1,2,3,4,5}.

Not Reflexive, symmetric, not antisymmeytic, transitive.

c) The complementary relation of R={(1,3) , (2, 3), (3, 1), (3,2) } on the set {1,2,3}.

Reflexive, symmetric, not antisymmeytic, transitive.

d) The composite of "less than" relation on the set of intergers and "greater than" relation on the set of intergers.

Reflexive, symmetric, not antisymmeytic, transitive.

Example: Let S={1,2,3,…}, then R1°R2= $\begin{vmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{vmatrix} \cdot \begin{vmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{vmatrix}$

2. [8 points] Use Warshall's algorithm to find the transitive closure of R on {1,2,3,4,5 } where R={(1, 2),(1,3),(2, 2), (2, 3) (2,4), (3, 2), (3, 3),

Example: Let S={1,2,3,…}, then R1°R2= $\begin{matrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{matrix} \cdot \begin{matrix} 1 & 0 & 0 \\ 1 & 1 & 0 \end{matrix} = \begin{matrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{matrix}$

2. [8 points] Use Warshall's algorithm to find the transitive closure of R on {1,2,3,4,5 } where R={(1, 2),(1,3),(2, 2), (2, 3), (2, 4), (3, 2), (3, 3), (3, 4),(3,5),(4,5) ,(5,1) }.

$$w0=\begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}, w1=\begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 \end{bmatrix}, w2=\begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix},$$

$$w3=\begin{bmatrix}0&1&1&1&1\\0&1&1&1&1\\0&1&1&1&1\\0&0&0&0&1\\1&1&1&1&1\end{bmatrix}, w4=w3,\ w5=\begin{bmatrix}1&1&1&1&1\\1&1&1&1&1\\1&1&1&1&1\\1&1&1&1&1\\1&1&1&1&1\end{bmatrix}$$

按 walshall 算法写出过程，w0 对得 2 分，w1 对得 2 分；最终结果 w5 全对得 4 分，w5 中错 2 个元素扣 1 分,扣完 4 分为止。

3. [8 points] Let R_1 is an equivalence relation produced by the partition A_1={a,b}, A_2={c,d},and A_3={e,f} of S={a,b,c,d,e,f}, R_2 is another equivalence relation produced by the partition A_1={a, c}, A_2={ b,d}, A_3={e},and A_4={ f} of S={a,b

3. [8 points] Let R_1 is an equivalence relation produced by the partition A_1={a,b}, A_2={c,d},and A_3={e,f} of S={a,b,c,d,e,f}, R_2 is another equivalence relation produced by the partition A_1={a, c}, A_2={ b,d}, A_3={e},and A_4={ f} of S={a,b,c,d,e,f}.

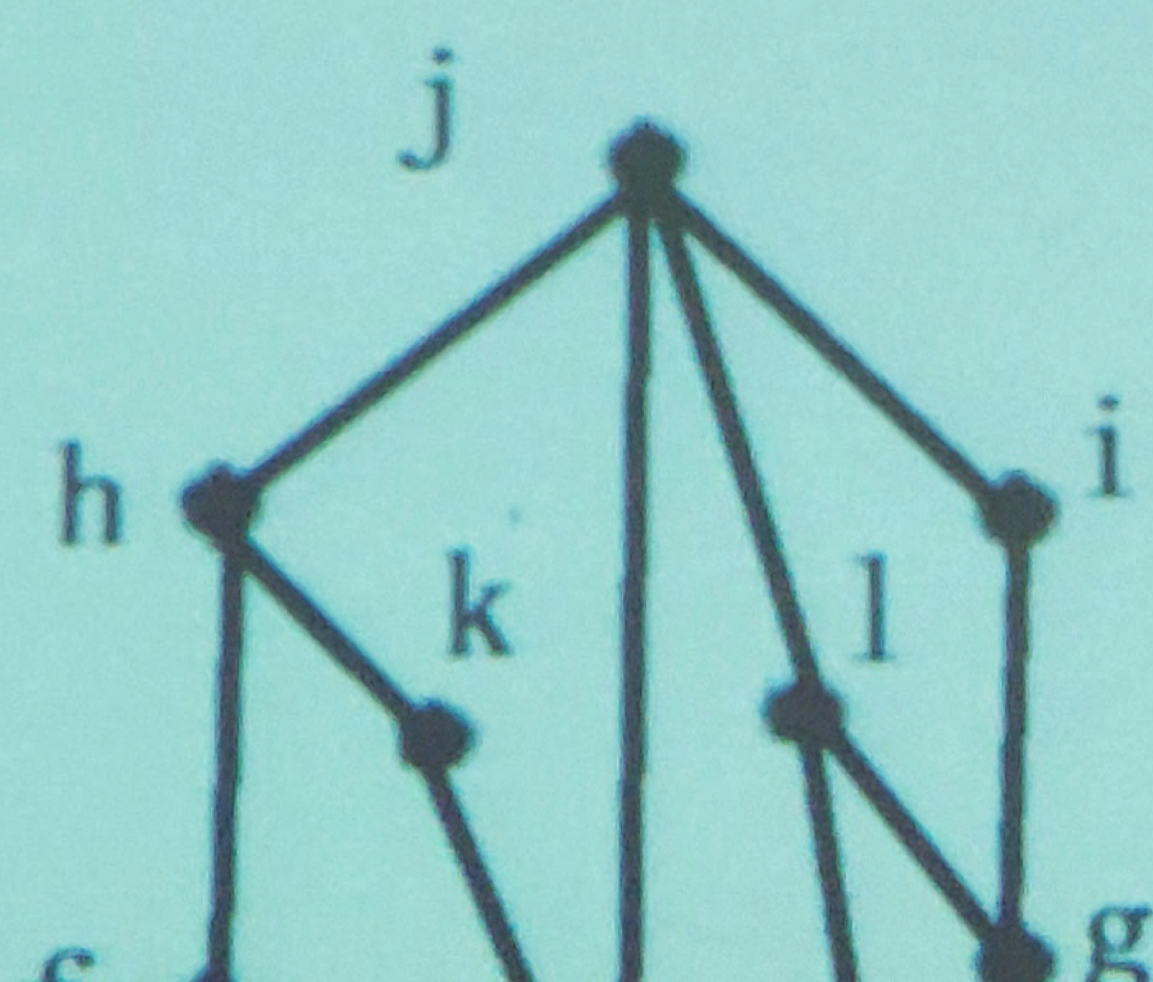
List the ordered pairs of relation (1) $R_1 \cap R_2$ and (2) $R_1 \oplus R_2$.

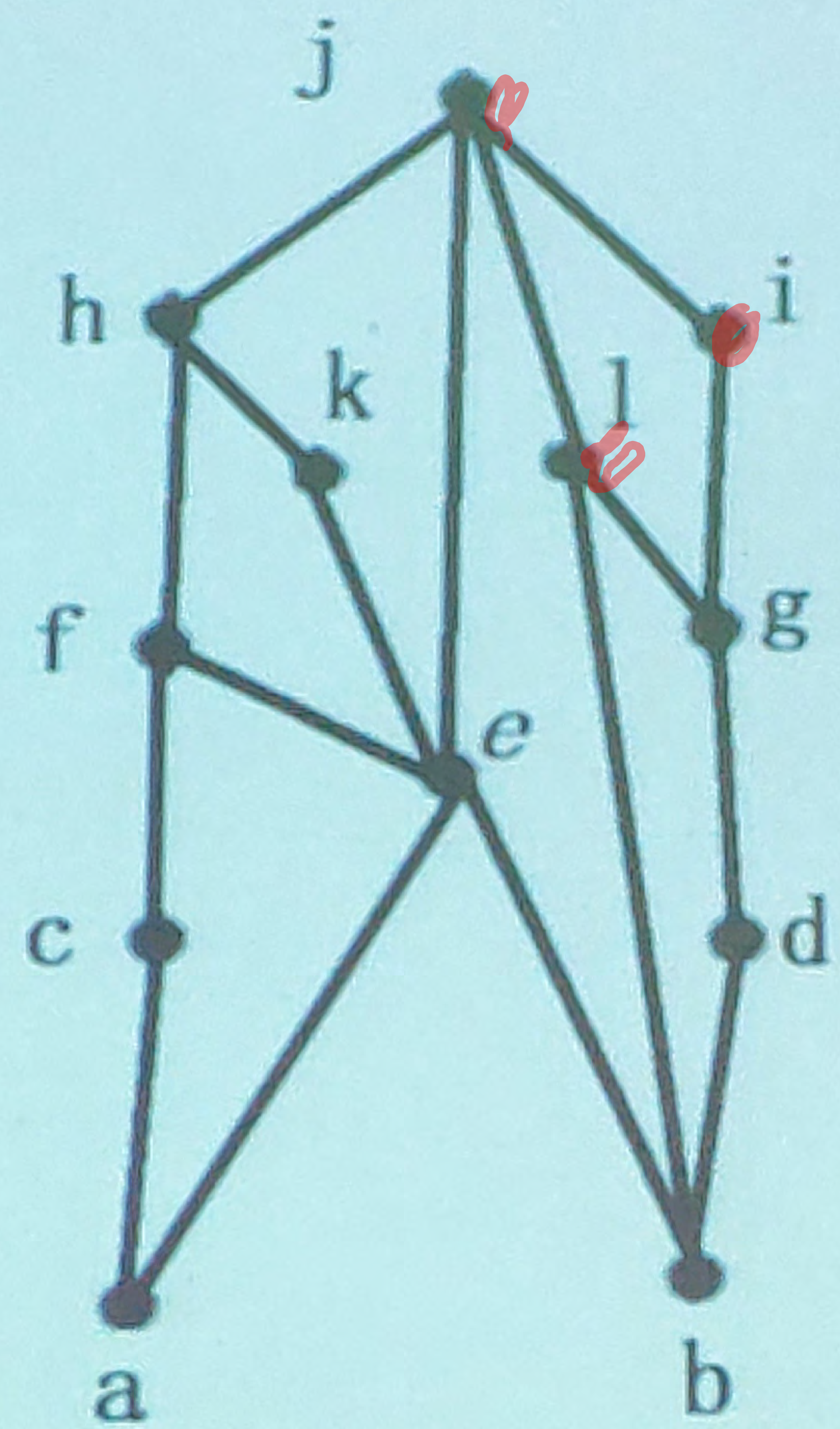
(1) R1 ∩R2 ={ (a,a), (b,b), (c,c), (d,d), (e,e), (f,f) }

(2) R1 ⊕R2 = { (a,b), (b,a), (c,d), (d,c), (e,f), (f,e), (a,c), (c,a), (b,d), (d,b) }.

每问4分，错2个元素扣1分,扣完4分为止。

4. [12 points] Answer these questions for the partial order represented by this Hasse diagram.





(1) Find all maximal elements.

{j} 正确得 1 分

(2) Find all minimal elements.

(4) Is there a least element?

No 正确得 1 分

(5) Find all upper bounds of {a, b, c}.

{ f , h, j } 正确得 2 分,写错或漏一个扣 1 分

(6) Find the least upper bound of {e, f, g}, if it exists.

j 正确得 2 分,答不存在的扣 2 分，写错元素的扣 1 分。

(7) Find all lower bounds of {h, j, k}.

{ k, e, a, b } 正确得 2 分，写错或漏一个扣 1 分.

(8) Find the greatest lower bound of {i, j,l}, if it exists.

g 正确得 2 分,答不存在的扣 2 分，写错元素的扣 1 分。

5. [10 points] Let Q be the set of rational numbers and define a*b=a+b-ab

(a) Is (Q,*) a monoid? Justify your answer.

(b) If (Q,*) a monoid, which elements of Q have an inverse?

答:（1）

由于 *a,b* 是实数，所以 *ab=a+b -ab* 也是实数，运算闭合。（1 分）

5. [10 points] Let Q be the set of rational numbers and define a*b=a+b-ab

(a) Is (Q,*) a monoid? Justify your answer.

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答:（1）

由于 a,b 是实数，所以 $ab=a+b-ab$ 也是实数，运算闭合。（1 分）

$$a * (b * c) = a * (b+c-b\times c) = a+(b+c-b\times c) - a\times(b+c-b\times c)$$
$$= a+b+c-bc-ab-ac+abc$$
$$(a * b) * c = (a+b-a\times b) * c = a+b-a\times b + c - (a+b-a\times b)\times c$$
$$= a+b+c-ab-ac-bc+abc$$

因此 *是可结合的。（3 分）

由于 $a * 0 = 0 * a = a$，所以 0 是单位元，(Q, *)是独异点。（1 分）

（2）

如果 $a\neq 1$,

$$a * \frac{-a}{1-a} = a + \frac{-a}{1-a} + a\times \frac{-a}{1-a} = 0 \qquad （3 分）$$

所以，Q 中除 1 以外都有逆元。（2 分）

6. [10 points] Let G be a group and K is a subgroup of G. Show that for any element a,b in G, either aK=bK or $aK \cap bK=\Phi$.

解法 1--

证明：思路是构造一个等价关系，使得 K 的每个左陪集都是其中的等价类。

1、构造一个二元关系 $R=\{aRb|a^{-1}*b\in K\}$。 （2 分）

下面证明它是一个等价关系。

自反性：$\forall x\in G, x^{-1}*x=e\in K, \Rightarrow xRx$.（1 分）

对称性：$\forall x,y\in G, xRy \Rightarrow x^{-1}*y\in H$, so $y^{-1}*x=(x^{-1}*y)^{-1}\in K, yRx$. （1 分）

传递性：$\forall x,y,z\in G$, (xRy and yRz)$\Rightarrow x^{-1}*y\in K$ and $y^{-1}*z\in K$.

$x^{-1}*z=x^{-1}*y*y^{-1}*z\in K, xRz$. （1 分）

2、证明 R 的等价类是 K 的左陪集。

解法 2--

证明:

假设 $aK \cap bK \neq \Phi$，设 $x \in aK \cap bK$ 则 $\exists k_1,k_2 \in K, x= ak_1=bk_2$,由此可得 $a=b\, k_2\, k_1^{-1}$（3 分）

对$\forall y \in aK$, $\exists k_3 \in K$, $y=a\, k_3= b\, k_2\, k_1^{-1}\, k_3$. 因 K 为 G 的子群，所以 $k_2\, k_1^{-1}\, k_3 \in K$

即$\exists h= k_2\, k_1^{-1}\, k_3 \in K$， $y=bh$,所以 $y \in bK$.即 $aK \subseteq bK$。（4 分）

同理可证，$bK \subseteq aK$.（2 分）

所以 $aK= bK$.（1 分）

7. [8 points] Let (G,*) be a group. Show that If $(a*b)^2=a^2*b^2$ for all a and b in G then G is Abelian.

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proof:

$(a*b)^2 = (a*b)*(a*b) = a*(b*a)*b$, for G is associative. （2 分）

$a^2*b^2 = (a*a)*(b*b) = a*(a*b)*b$, for G is associative. （2 分）

based left/right cancellation, $b*a=a*b$. （4 分）

Hence, G is Abelian.

8. [12 points] Let Z be the set of all integers and let + be the binary operation o addition on Z. Let $B = \{0, 1\}$, and let $+_2$ be the operation defined on B as follow

$+_2$	0	1
0	0	1
1	1	0

. Let f be a function from group $(Z, +)$ to $(B, +_2)$.

(a) Prove that f defined by $f(x)=x(\text{mod } 2)$ is homomorphism from Z to B.

证：（4 分）

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. Let f be a function from group $(Z, +)$ to $(B, +_2)$.

(a) Prove that f defined by $f(x)=x(\text{mod } 2)$ is homomorphism from Z to B.

证：（4 分）

$f(x+y)= (x+y)(\text{mod } 2)= (x(\text{mod } 2)+ y(\text{mod } 2))\text{mod } 2=(f(x)+ f(y)) \text{ mod } 2$

$= f(x)+_2 f(y)$

所以 f defined by $f(x)=x(\text{mod } 2)$ is homomorphism from Z to B

(b) Find ker(f). （4 分）

$\ker(f)=\{2x|x\in Z\}$

ker(f)={2x|x∈Z}

(c) Write the operation table of quotient group $Z/\ker(f)$. （4 分）

⊕	[0]	[1]
[0]	[0]	[1]
[1]	[1]	[0]

（4 分，运算表正确可得分，错一个扣 1 分，扣完为止）

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$

9. [8 points] Let $H = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ be a parity check matrix.

(a) Determine the (3,7) group code e_H:$B^3 \rightarrow B^7$.

（6分， 8个结果错一个扣 1分，扣完为止）

$$B^3 * H = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} * \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}$$

或者写成如下形式：

9. [8 points] Consider the (3,6) group encoding function e: $B^3 \to B^6$ defined by

e(000)=000000, e(001)=001011, e(010)=010101,
e(011)=011110, e(100)=100110, e(101)=101101,
e(110)=110011, e(111)=111000.

(a) How many errors will (e, d) correct?

Because the minimal distance of e is 3，(e, d) can correct 1 error.
(2分,答对结果数即可得分,结果错但最小距离算对了可得1分)

10. [8 points] Let m=3, n=6, H= $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ be a parity check matrix.

(a) Compute the syndrome for the coset leaders { 000000，000001, 000010, 000100, 001000, 010000, 100000, 000110} for $N=e_H(B^3)$.

（5分， 8对结果错一个扣 0.5分，最终扣分若有小数则向下取整）

Syndrome	Coset leader
000	000000
001	000001
010	000010
100	000100
011	001000
101	010000
111	100000

function associated with e_H by using the coset leaders and their syndromes from (a).

a)100110 b) 011011 c) 110001

（3分，错一个结果扣1分。）

d(100110)=100, d(011011)=001, d(110001)=111.

Because 100110*H=001, so coset leader is 000001,
100110⊕000001=100111, so b=100.
Similar, 011011*H=101, so coset leader is 010000,
011011⊕010000=001011, so b=001.
110001*H=011, so coset leader is 001000, 110001⊕001000=111001, so b=111.